Patch Locale of a Spectral Locale in Univalent Type Theory

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What is a locale?

Notion of space characterised solely by its frame of opens.

Spectral and Stone locales

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Stone

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Every Stone locale is spectral.

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Every Stone locale is spectral.

Patch universally transforms a spectral locale into a Stone one.

Patch as a coreflector

$$\mathbf{Stone} \underset{Patch}{\overset{i}{\underset{\perp}{\rightleftarrows}}} \mathbf{Spec}$$

Some examples of patch

Spectral locale in consideration **Its patch**

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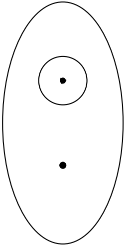
Sierpiński space (Ω)



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$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$

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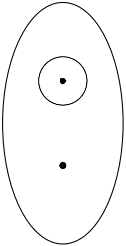	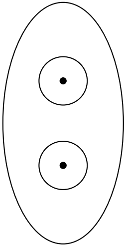
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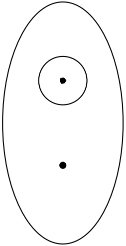	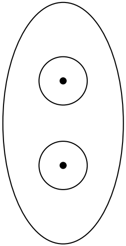
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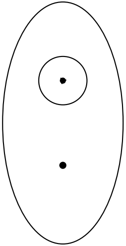
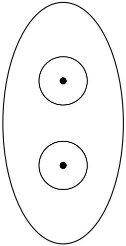
Sierpiński space (Ω) 	Booleans ($\mathbf{2}$) 
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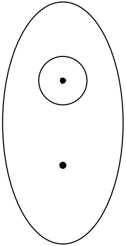
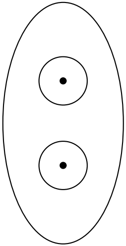
Sierpiński space (Ω) 	Booleans ($\mathbf{2}$) 
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Goal

Implement the patch locale in univalent type theory predicatively i.e. without using propositional resizing axioms.

Type-theoretical preliminaries and notation

Definition (Family)

The type of $\mathcal{W}$-families over a type $A : \mathcal{U}$ is $\mathsf{Fam}_{\mathcal{W}}(A) :\equiv \Sigma_{I:\mathcal{W}} (I \to A)$.

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Definition (Propositional resizing)

The propositional resizing axiom from $\mathcal{U}$ to $\mathcal{V}$ says

$$\prod_{P:\mathcal{U}} \text{is-prop}(P) \to \text{is-}\mathcal{V}\text{-small}(P).$$

Frames in type theory

Definition (Frame)

A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-*frame* consists of

- a type $A : \mathcal{U}$,
- a partial order $- \leq - : A \to A \to \mathrm{hProp}_{\mathcal{V}}$,
- a top element $\top : A$,
- a binary meet operation $- \wedge - : A \to A \to A$,
- a join operation $\bigvee \_ : \mathrm{Fam}_{\mathcal{W}}(A) \to A$,
- satisfying distributivity i.e. $x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$ for every $x : A$ and family $\{y_i\}_{i:I}$ in A.

Some notation

A frame homomorphism is a function preserving finite meets and arbitrary joins.

The category of frames and their homomorphisms is denoted **Frm**; its opposite is denoted **Loc**.

- Morphisms of **Loc** are called continuous maps.

The frame corresponding to a locale X is denoted $\mathcal{O}(X)$.

We work in the spatial direction:

- $X, Y, Z, \ldots$ range over locales;
- $f, g : X \to Y$ range over continuous maps;
- $U, V, W, \ldots : \mathcal{O}(X)$ range over opens; and
- $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ denotes the frame homomorphism corresponding to a continuous map $f : X \to Y$ of locales.

Notions of size for locales

The generality of $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locales is **almost never** needed.

Definition

A small locale is a $(\mathcal{U}, \mathcal{U}, \mathcal{U})$-locale.
A large and locally small locale is a $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-locale.

Impredicatively, one works with small locales.

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Theorem (de Jong and Escardó [6] [5])

The existence of a nontrivial small locale implies a form of propositional resizing.

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Theorem (de Jong and Escardó [6] [5])

The existence of a nontrivial small locale implies a form of propositional resizing.

Most locales that arise in practice in our setting are *large* *and* *locally small*.

Bases for locales

Definition (Basis)

A $\mathcal{W}$-family $\{B_i\}_{i:I}$ over a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locale X is said to form a basis for X if

for any $U : \mathcal{O}(X)$, there is a *subfamily* $\{B_l\}_{l \in L}$ of $\{B_i\}_{i:I}$ such that $U = \bigvee_{l \in L} B_l$.

Note that a locale basis can be assumed WLOG to be directed.

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In our work, we focus on:

large and locally small frames with small bases.

A description of Patch

A nucleus on frame L is an endofunction $j : |L| \to |L|$ that is inflationary, idempotent, and preserves binary meets.

A nucleus is called Scott-continuous if it preserves joins of directed families.

Patch is the frame of Scott-continuous nuclei.

Previous work [3] exploited the fact that Patch is a subframe of the frame of all nuclei.

Joins in the frame of *all* nuclei (1)

Consider a locale X and let $j, k : \mathcal{O}(X) \to \mathcal{O}(X)$ be two nuclei.

Ordering: $j \preceq k :\equiv \prod_{U:\mathcal{O}(X)} j(U) \leq k(U)$

Top element: $\mathbf{1} :\equiv U \mapsto \mathbf{1}_X$.

Binary meets: $j \curlywedge k :\equiv U \mapsto j(U) \wedge k(U)$.

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Unfortunately, the pointwise join fails to be idempotent in general.

Joins in the frame of *all* nuclei (2)

It is possible to construct the joins in the frame of all nuclei impredicatively.

Previous constructions include those by

- Simmons [7],
- Banaschewski [1],
- Johnstone [4],
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All of these use some form of impredicativity. Moreover, we are not aware of any predicative construction of the frame of all nuclei.

The frame of Scott-continuous nuclei in type theory?

Question

Can this patch construction be shown to be the coreflector in the predicative setting of univalent type theory?

The frame of Scott-continuous nuclei in type theory?

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Can this patch construction be shown to be the coreflector in the predicative setting of univalent type theory?

Our answer: **Yes**, if one considers large and locally small frames with small bases.

Joins in the frame of Scott-continuous nuclei (1)

Let $K :\equiv \{k_i\}_{i:I}$ be a family of nuclei.

Finite compositions of nuclei

Denote by K^* the family of all finite compositions of k_i's:

$$K^* :\equiv (\mathsf{List}(I), \beta)$$

where $\beta(i_0, \ldots, i_{n-1}) :\equiv k_{i_{n-1}} \circ \cdots \circ k_{i_0}$.

Proposition

The family K^* is always directed.

Joins in the frame of Scott-continuous nuclei (2)

When only the Scott-continuous nuclei are considered, however, this join exists predicatively!

Let $K :\equiv \{k_i\}_{i:I}$ be a family of Scott-continuous nuclei on locale X and let $U : \mathcal{O}(X)$.

$$\left(\bigvee_{i:I} k_i\right)\left(\bigvee_{i:I} k_i\right)(U) \equiv$$

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Compactness

Definition (Compact open)

Let $U : \mathcal{O}(X)$ be an open of some locale X. The open U is called compact if

$$U \leq \bigvee_{i:I} V_i \rightarrow \exists k : I.\, U \leq V_k$$

for every *directed* family of opens $\{V_i\} : \mathsf{Fam}_{\mathcal{W}}(\mathcal{O}(X))$.

Definition (Compact locale)

A locale X is called compact if its top open is compact.

The type-theoretical construction of Patch

Recall the definition of a spectral locale:

> *a locale in which the compact opens form a basis closed under finite meets.*

One can predicatively construct the patch locale on an arbitrary locale.

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One can predicatively construct the patch locale on an arbitrary locale.

The need for a small basis arises in multiple places:

- to ensure that Patch is locally small,
- to ensure the existence of open nuclei (to be explained later).

Ordering on nuclei – *size matters*

Let

- X be a large and locally small locale, and
- j and k be two Scott-continuous nuclei on X.

Define $j \preceq k :\equiv \prod_{U:\mathcal{O}(X)} j(U) \leq k(U)$.

Problem: $j \preceq k$ lives in universe $\mathcal{U}^+$.

This means Patch(X) is a $(\mathcal{U}^+, \mathcal{U}^+, \mathcal{U})$-locale i.e. it is *not* locally small.

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Definition

$j \preceq_S k :\equiv \prod_{C:\mathcal{K}(X)} j(C) \leq k(C)$

Closed and open nuclei

We embed the opens of X into Patch(X) using the closed and open nuclei.

- Open nucleus induced by U:
  - Represents the open subspace corresponding to U;
  - Defined as $\neg`U' :\equiv V \mapsto U \Rightarrow V$.
- Closed nucleus induced by U,
  - Represents the closed subspace corresponding to the complement of U;
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Heyting implication is defined using the Adjoint Functor Theorem which amounts to the definition:

$$U \Rightarrow V :\equiv \bigvee \{W : \mathcal{O}(X) \mid W \wedge U \leq V\}.$$

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Spectrality revisited (1)

To solve these problems, we start with a spectral locale X assumed to

- be large and locally small;
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We make exactly the same assumptions for zero-dimensionality, involved in the definition of a Stone locale.

Spectrality revisited (2)

Proposition

In any spectral locale with small basis, any compact open falls in the basis.

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Proof

- Let $C : \mathcal{O}(X)$ be a compact open.
- $C = \bigvee_{l \in L} B_l$ for some basic covering family $\{B_l\}_{l \in L}$.
- By compactness, there exists an index $k \in L$ such that $C \leq B_k$.
- Clearly, $B_k \leq C$ which means $B_k = C$.

Spectrality revisited (3)

We can do better! Let X be a spectral locale with small basis $\{B_i\}_{i:I}$.

Theorem

There is an isomorphism

$$\left(\Sigma_{U:\mathcal{O}(X)} \exists_{i:I} U = B_i\right) \quad \simeq \quad \Sigma_{U:\mathcal{O}(X)} \,\text{is-compact-open}(U).$$

Proposition

Assuming the existence of quotients, the type $\Sigma_{U:\mathcal{O}(X)} \exists_{i:I} U = B_i$ is $\mathcal{U}$-small.

Corollary

The type of compact opens of X is $\mathcal{U}$-small.

Patch is Stone

Theorem

Given a spectral locale A with small basis $\{B_i\}_{i:I}$, Patch(A) is a Stone locale (with a small basis consisting of clopens).

Proof sketch

The family

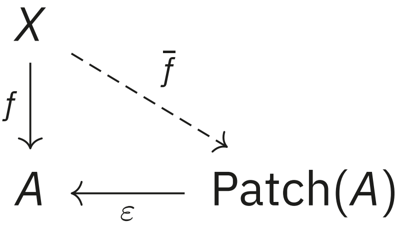
$$\{`B_m\text{'} \wedge \neg `B_n\text{'} \mid m, n : I\}$$

forms a basis for Patch(A) and the covering subfamily for a given Scott-continuous nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ is given by

$$\{`B_m\text{'} \wedge \neg `B_n\text{'} \mid B_m \leq j(B_n), m, n : I\}.$$

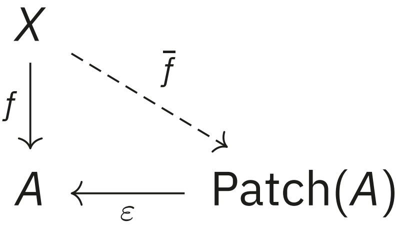
The universal property of Patch

Let A and X be, respectively, a spectral and a Stone locale with small bases. Denote by $\{B_i\}_{i:I}$ the small basis of A.



The universal property of Patch

Let A and X be, respectively, a spectral and a Stone locale with small bases. Denote by $\{B_i\}_{i:I}$ the small basis of A.



$$\bar{f}^* : \mathcal{O}(\mathsf{Patch}(A)) \to \mathcal{O}(X)$$

$$\bar{f}^* :\equiv j \mapsto \bigvee_{m,n:I} \{f^*(B_m) \wedge \neg f^*(B_n) \mid B_m \leq j(B_n)\}$$

Summary

We set out to implement a predicative version of an impredicative construction from pointfree topology in univalent type theory.

Doing this predicatively turned out to involve surprising challenges.

We had to reformulate quite a few things in the theory itself to obtain a type-theoretic understanding of the construction in consideration.

I have almost completely formalised our work in the Agda proof assistant, as part of Escardó's `TypeTopology` [0] library.

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