Predicative Stone Duality in Univalent Foundations

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What is Stone duality?

Stone spaces	$\simeq$	Boolean algebras
Spectral spaces	$\simeq$	Distributive lattices

- Stone [Sto36] first discovered this duality in the context of Boolean algebras.
- He then generalized [Sto37] it to distributive lattices.

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What is Stone duality?

Let X be a **Stone space**.

The clopens of X form a **Boolean algebra**.

Let L be a **Boolean algebra**.

The ultrafilters of L form a **Stone space**.

In Stone spaces, the **clopens** coincide with the **compact opens**.
It is really the *algebra of compact opens* that matters!

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This talk

We develop the Stone duality between **spectral spaces** and **distributive lattices** in the foundational setting of

univalent type theory,

which is

constructive and predicative

by default.

Stone duality is classical in a point-set setting.

Taking locales as our notion of **space**, however, Stone duality can be carried out in a *completely constructive* way.

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A **locale** is a notion of space defined solely by its lattice of opens.

In point-free topology, we do not *require* that a topology be a sublattice of the powerset lattice of *some set of points*.

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Foundations

Notions of size

Definition ($\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **$\mathcal{V}$-small** if it has a copy in universe $\mathcal{V}$ i.e.

$$\Sigma_{(Y : \mathcal{V})} X \simeq Y.$$

Definition (Local $\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **locally $\mathcal{V}$-small** if the identity type $x = y$ is $\mathcal{V}$-small for every pair of inhabitants $x, y : X$.

Definition (Ω)

We denote by $\Omega_{\mathcal{U}}$ the type of propositions in universe $\mathcal{U}$.

Definition (Propositional resizing)

The **propositional** $(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that every proposition $P : \Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.

Definition (Ω-resizing)

The Ω-$(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that $\Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.

Forms of resizing

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Proposition

LEM implies Ω-$(\mathcal{U}, \mathcal{V})$-resizing for all universes.

Proof sketch

- If LEM holds, all propositions are decidable i.e. $\Omega \simeq \mathbf{2}$.
- The type $\mathbf{2}$ always has a copy in $\mathcal{U}_0$.
- Types in $\mathcal{U}_0$ can always be lifted up to any universe.

Predicative mathematics is a branch of constructive mathematics.

More on resizing

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Univalence and the propositionality of being small

Definition (Univalence)

A universe $\mathcal{U}$ is called **univalent** if, for every pair of types $X, Y : \mathcal{U}$, the map $\mathrm{idtoeqv} : X =_{\mathcal{U}} Y \to X \simeq Y$ is an equivalence.

Definition (The univalence axiom)

The **univalence axiom** says that *every* universe is univalent.

Generalization of Propositions 2.8 and 2.9 of [dJE23]

The following are equivalent:

- For every type $A : \mathcal{U}$, the type expressing that A is $\mathcal{V}$-small is a proposition (for every pair of universes $\mathcal{U}$ and $\mathcal{V}$).
- The univalence axiom holds.

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Basics of point-free topology

Definition (Frame)

A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$**-frame** consists of

- a type $A : \mathcal{U}$,
- a partial order $\_ \leq \_ : A \to A \to \Omega_{\mathcal{V}}$,
- a top element $\mathbf{1} : A$,
- a binary meet operation $\_ \wedge \_ : A \to A \to A$,
- a join operation $\bigvee \_ : \mathsf{Fam}_{\mathcal{W}}(A) \to A$;
- satisfying distributivity i.e. $x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$ for every $x : A$ and $\mathcal{W}$-family $(y_i)_{i:I}$ in A.

Large, **locally small**, and **small-complete frame**: $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-frame.

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No-go theorem for complete, small lattices

Curi [Cur10] previously showed:

> *CZF cannot prove that certain classes of nontrivial complete lattices (including join-lattices, dcpos, and frames) are small.*

He achieves this by showing that CZF is consistent with an anti-classical principle called Generalized Uniformity Principle (GUP).

de Jong and Escardó gave an analogous result [dJE23] in the style of **reverse constructive mathematics** [Ish06].

They show **directly** that certain results cannot be obtained predicatively, by deriving resizing axioms from them.

Theorem

If there exists a nontrivial small frame then Ω-resizing holds.

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The right category of frames

A predicative investigation of locale theory in HoTT/UF **must** focus on large and small-complete frames.

For us, **frame** means $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-frame
(over some base universe $\mathcal{U}$).

$\mathbf{Frm}_{\mathcal{U}}$: the category of such frames and frame homomorphisms.
$\mathbf{Loc}_{\mathcal{U}}$: the opposite of this category.

We denote by $\mathcal{O}(X)$ the frame defining a locale X.
A continuous map $f : X \to Y$ of locales is given by a homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

Definition (Weak base)

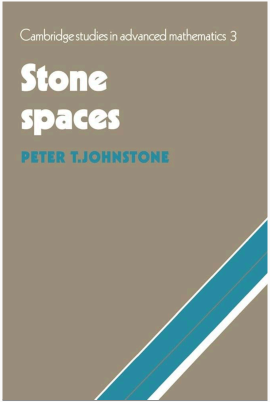
A family $(B_i)_{i:I}$ of opens forms a **weak base** for locale X if

for every $U : \mathcal{O}(X)$, there is an unspecified, directed, small family $(i_j)_{j:J}$ on the base index satisfying $U = \bigvee_{j:J} B_{i_j}$.

Compact and spectral locales

What is a spectral locale?

To write the definition of spectral locale in HoTT/UF, we look it up in a standard textbook...



We define a locale A to be *coherent* if

(i) Every element of A is expressible as a join of finite elements, and

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(ii) The finite elements form a sublattice of A – equivalently (by the Lemma), 1 is finite, and the meet of two finite elements is finite.

Compactness

Definition (Compact open)

An open $U : \mathcal{O}(X)$ is called **compact** if for every directed family $(V_i)_{i:I}$ with $U \leq \bigvee_{i:I} V_i$, there is some $k : I$ such that $U \leq V_k$.

Same as the "covers have finite subcovers" definition but with Kuratowski finiteness.

We define $\mathsf{K}^+(X) :\equiv \Sigma_{(U : \mathcal{O}(X))}$ is-compact(U).

- Observe that this type is **large** i.e. lives in $\mathcal{U}^+$.

Definition (Compact locale)

A **compact locale** is one in which the top open $\mathbf{1}$ is compact.

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Definition (Spectral locale) [Tos25; AET24]

A locale X is called **spectral** if it satisfies the following conditions:

- **(SP1)** It is compact (i.e. the empty meet is compact).
- **(SP2)** Compact opens are closed under binary meets.
- **(SP3)** The type $\mathsf{K}^+(X)$ forms a weak base.
- **(SP4)** The type $\mathsf{K}^+(X)$ is small.

→ A continuous map $f : X \to Y$ is called **spectral** if $f^*(K)$ is a compact open of Y for every compact open K of X.

Spec$_{\mathcal{U}}$: the category of spectral locales and spectral maps.

Lemma

Univalence implies that being spectral is a proposition.

Univalence seems to be required to write down
the property expressing the mathematical notion in consideration!

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Small distributive lattices

Definition (Small distributive lattice)

A **distributive $\mathcal{U}$-lattice** consists of

- a set $|L| : \mathcal{U}$,
- elements $\mathbf{0}, \mathbf{1} : |L|$,
- operations $- \wedge - : |L| \to |L| \to |L|$ and $- \vee - : |L| \to |L| \to |L|$,
- satisfying the laws of **associativity**, **commutativity**, **unitality**, **idempotence**, and **absorption**.

$\mathbf{DLat}_{\mathcal{U}}$: the category of distributive $\mathcal{U}$-lattices or small distributive lattices.

Definition (Ideal)

A $\mathcal{U}$-**ideal** of a distributive lattice L is a subset $I : L \to \Omega_{\mathcal{U}}$ satisfying the conditions:

- inhabitedness,
- downward closedness,
- closedness under binary joins.

Spectrum construction: $\mathsf{DLat}_{\mathcal{U}} \to \mathsf{Spec}_{\mathcal{U}}$

Lemma

For every distributive $\mathcal{U}$-lattice L, the type $\mathsf{Idl}_{\mathcal{U}}(L)$ forms a frame i.e. a **large**, **locally small**, and **small-complete** frame.

Lemma

For every distributive $\mathcal{U}$-lattice L, the frame $\mathsf{Idl}_{\mathcal{U}}(L)$ is spectral.

The **spectrum of** L is the locale defined by $\mathsf{Idl}_{\mathcal{U}}(L)$.

- We denote this by $\mathsf{Spec}(L)$.

Observe the following:

- Classically: we work with the ideals $L \to \mathbf{2}$.
- Constructively but impredicatively: we work with the ideals $L \to \Omega$.
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Small distributive lattice of compact opens: $\mathrm{Spec}_{\mathcal{U}} \to \mathrm{DLat}_{\mathcal{U}}$

Recall that the type $\mathsf{K}^+(X)$ is *a priori* large — it lives in $\mathcal{U}^+$.

- In other words, it falls in the category $\mathrm{DLat}_{\mathcal{U}^+}$ and not $\mathrm{DLat}_{\mathcal{U}}$.

Condition **(SP4)** gives us a specified, small type X_0 such that

$$\mathsf{K}^+(X) \simeq X_0.$$

Lemma

For every $\mathcal{V}$-distributive lattice L, if the carrier set $|L|$ is $\mathcal{U}$-small then L has a copy in $\mathcal{U}$ i.e. is isomorphic to a specified $\mathcal{U}$-distributive lattice.

- We just transport the lattice structure through the equivalence, which is always a lattice isomorphism.

Lemma

For every spectral locale X, we have a specified, small distributive lattice $\mathsf{K}(X)$.

Predicative Stone duality — object part

We have thus constructed maps:

$$\mathbf{Spec}_{\mathcal{U}} \underset{\mathsf{Spec}}{\overset{\mathsf{K}}{\rightleftarrows}} \mathbf{DLat}_{\mathcal{U}}$$

Proposition

Assuming univalence (twice), the maps K and Spec form a type equivalence.

- We thus have an equivalence $\mathbf{Spec}_{\mathcal{U}} \simeq \mathbf{DLat}_{\mathcal{U}}$.
- Observe that $\mathbf{Spec}_{\mathcal{U}} : \mathcal{U}^{++}$ and $\mathbf{DLat}_{\mathcal{U}} : \mathcal{U}^{+}$, but the result says $\mathbf{Spec}_{\mathcal{U}}$ is $\mathcal{U}^{+}$-small.

Recall that a spectral map is a continuous function $f : X \to Y$ such that

$$\Pi_{(V : \mathcal{O}(Y))} V \text{ is compact} \to f^*(K) \text{ is compact.}$$

This is a mapping $\mathsf{K}^+(Y) \to \mathsf{K}^+(X)$.

We define maps:

- $\mathsf{K} : \mathsf{Hom}(X, Y) \to \mathsf{Hom}(\mathsf{K}(Y), \mathsf{K}(X))$
- $\mathsf{Spec} : \mathsf{Hom}(K, L) \to \mathsf{Hom}(\mathsf{Spec}(L), \mathsf{Spec}(K))$

Theorem

The above functors form a categorical equivalence.

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Predicative Stone duality — morphisms

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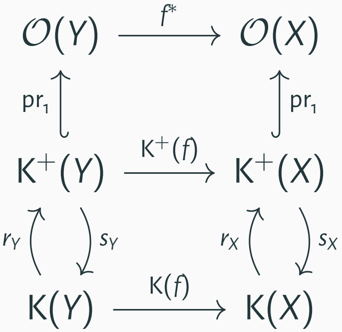

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Conclusion and further work

- We can obtain a predicative form of Stone duality by considering $\Omega_{\mathcal{U}}$-valued ideals on $\mathcal{U}$-small lattices.
- Alternative to formal topology.
- This fits well into our investigation of predicative locale theory in the category of **large**, **locally small**, and **small-complete** locales.
- Completely formalized in AGDA as part of TYPETOPOLOGY.
- TODO: Patch of a spectral locale should give the **free Boolean extension** of a distributive lattice.
  - Constructed in previous work [AET24].
- TODO: Constructive and predicative Priestley duality.
- TODO: Spectrum of a commutative ring?
- TODO: Further investigation of links with Tom de Jong's doctoral work on domain theory [dJon23].
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