Formal Topology in Univalent Foundations

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What is topology?

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A topological space is a set X together with a collection $\Omega(X)$ of its subsets such that

- $\Omega(X)$ is closed under finite intersections, and
- $\Omega(X)$ is closed under arbitrary unions.

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Let P be a program. When run, it prints a sequence of integers.

We observe that its output starts as:

7 11 2 2 8 42 . . .

We can consider certain properties of P, such as:

"P eventually prints 17", or

"P prints no more than two 2s".

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Observable

Not observable

What is topology?

“ϕ is an observable property.”

$$\leftrightarrow$$

If a program satisfies ϕ, there exists a stage *m* of the output σ at which the program is verified to satisfy ϕ: all extensions of $\sigma|_m$ satisfy ϕ.

What is topology?

Let $\phi_1, \cdots, \phi_n$ be a finite number of observable properties.

Suppose $\phi_1 \wedge \cdots \wedge \phi_n$ holds.

There must be stages $m_1, \cdots, m_n$ such that ϕ_k is verified at m_k.

$\phi_1 \wedge \cdots \wedge \phi_n$ must then be verified at $\max(m_1, \cdots, m_n)$.

If $\phi_1, \cdots, \phi_n$ are observable then so is $\phi_1 \wedge \cdots \wedge \phi_n$.

What is topology?

Let $\{ \psi_i \mid i \in I \}$ be an arbitrary number of observable properties.

Suppose $\bigvee_i \psi_i$ holds.

Some ψ_i holds meaning it must be verified at some stage m.

$\bigvee_i \psi_i$ is hence verified at stage m.

If $\{ \psi_i \mid i \in I \}$ are observable then so is $\bigvee_i \psi_i$.

Topology is a mathematical theory of observable properties.[1]

[1] as pointed out by Scott [5], Smyth [6], Abramsky [1], Vickers [9], Escardó [2], and Taylor [7], among others. My presentation here follows specifically Smyth [6].

Frames

Frames

A frame is a poset $\mathcal{O}$ such that

- finite subsets of $\mathcal{O}$ have meets,
- arbitrary subsets of $\mathcal{O}$ have joins, and
- binary meets distribute over arbitrary joins:

$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} \left(a \wedge b_i \right),$$

for any $a \in \mathcal{O}$ and family $\{b_i \mid i \in I\}$ over $\mathcal{O}$.

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In type theory, the quantification over arbitrary subsets is problematic.

Frames — a prime example

Given a poset

$$\begin{aligned} A &: \mathsf{Type}_m \\ \sqsubseteq &: A \to A \to \mathsf{hProp}_m \end{aligned}$$

the type of downwards-closed subsets of A is:

$$\sum_{(U \,:\, \mathcal{P}(A))} \prod_{(x\ y \,:\, A)} x \in U \to y \sqsubseteq x \to y \in U,$$

where

$$\begin{aligned} \mathcal{P} &: \mathsf{Type}_m \to \mathsf{Type}_{m+1} \\ \mathcal{P}(X) &:\equiv X \to \mathsf{hProp}_m. \end{aligned}$$

Frames — a prime example

This forms a frame defined as:

$$
\begin{aligned}
\top &:\equiv \lambda\_.\ \mathsf{Unit} \\
A \wedge B &:\equiv \lambda x.\ (x \in A) \times (x \in B) \\
\bigvee_{i\,:\,I} B_i &:\equiv \lambda x.\ \left\| \sum_{(i\,:\,I)} x \in B_i \right\|.
\end{aligned}
$$

Frames — a prime example

Question: can we get any frame out of a poset in this way?

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One way is to employ the notion of a nucleus on a frame.

For this, we need to enrich the notion of a poset with a structure that gives rise to an appropriate nucleus (on its frame of downwards-closed subsets).

That structure is a formal topology.

Formal Topology

Formal Topologies — as Interaction Systems

An interaction structure [4] on some type A comprises three functions:

$$
\begin{aligned}
B &: A \to \mathrm{Type} && (1),\\
C &: \prod_{(a\,:\,A)} B(a) \to \mathrm{Type} && (2), \text{and}\\
d &: \prod_{(a\,:\,A)} \prod_{(b\,:\,B(a))} C(a,b) \to A && (3).
\end{aligned}
$$

An interaction system is a type A equipped with an interaction structure.

Formal Topologies — as Interaction Systems

A formal topology is an interaction system (B, C, d) on some poset P that satisfies the following two conditions.

1. **Monotonicity**:

$$\prod_{(a\,:\,A)} \prod_{(b\,:\,B(a))} \prod_{(c\,:\,C(a,b))} d(a, b, c) \sqsubseteq a.$$

2. **Simulation**:

$$\prod_{(a'\ a\,:\,A)} a' \sqsubseteq a \to$$

$$\prod_{(b\,:\,B(a))} \sum_{(b'\,:\,B(a'))} \prod_{(c'\,:\,C(a',b'))} \sum_{(c\,:\,C(a,b))} d(a', b', c') \sqsubseteq d(a, b, c).$$

Nuclei

A nucleus on a frame F is an endofunction $\mathbf{j} : |F| \to |F|$ such that:

$$
\begin{aligned}
\prod_{(x\ y\ :\ |F|)} \mathbf{j}\,(x \wedge y) &= \mathbf{j}\,(x) \wedge \mathbf{j}\,(y) && \text{[meet preservation],} \\
\prod_{(x\ :\ |F|)} x &\sqsubseteq \mathbf{j}\,(x) && \text{[inflation], and} \\
\prod_{(x\ :\ |F|)} \mathbf{j}\,(\mathbf{j}\,(x)) &\sqsubseteq \mathbf{j}\,(x) && \text{[idempotence].}
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$$\prod_{(x\ :\ |F|)} x \sqsubseteq \mathbf{j}(x) \qquad \text{[inflation], and}$$

$$\prod_{(x\ :\ |F|)} \mathbf{j}(\mathbf{j}(x)) \sqsubseteq \mathbf{j}(x) \qquad \text{[idempotence].}$$

This is a meet-preserving, idempotent monad!

Nuclei

Let F be a frame, and $\mathbf{j} : |F| \to |F|$ a nucleus on it.

The set

$$\sum_{(x \,:\, |F|)} \mathbf{j}\,(x) = x$$

of fixed points for $\mathbf{j}$ is itself a frame:

$$\begin{aligned} \top \quad &:\equiv \quad \top_F \\ \_ \wedge \_ \quad &:\equiv \quad \_ \wedge_F \_ \\ \bigvee_i x_i \quad &:\equiv \quad \mathbf{j}\left(\bigvee_i^F x_i\right). \end{aligned}$$

We denote this $\mathfrak{fix}\,(F, \mathbf{j})$.

Nuclei

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In the posetal case, our *modality* will be the covering relation induced by the structure of a formal topology.

The covering nucleus — naive attempt

Let

- $\mathcal{F}$ be a formal topology with underlying poset P,
- $a : |P|$, and
- $U : \mathcal{P}(|P|)$, a downwards-closed subset of P.

$a \lhd U$ is inductively defined via two rules.

$$\frac{a \in U}{a \lhd U}\ \text{dir} \qquad \frac{b : B(a) \qquad \prod_{(c\ :\ C(a,b))} d(a,b,c) \lhd U}{a \lhd U}\ \text{branch}$$

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*Notice: $a \lhd U$ is a **structure** and not a **property**.*

The covering nucleus — naive attempt

$\triangleright$ could be shown to be a nucleus, **if it had the type**

$$\begin{aligned} \lhd \quad &: \quad |P| \to \mathcal{P}(|P|) \to \mathrm{hProp} \\ \rhd \quad &: \quad \mathcal{P}(|P|) \to \mathcal{P}(|P|), \end{aligned}$$

but its type is

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Idea: use propositional truncation:

$$\begin{aligned} \|\_ \lhd \_\| &: |P| \to \mathcal{P}(|P|) \to \text{hProp} \\ \|\_ \rhd \_\| &: \mathcal{P}(|P|) \to \mathcal{P}(|P|)\,. \end{aligned}$$

The covering nucleus — naive attempt

Need to show: $\|\_ \lhd \_\|$ is a nucleus.

This involves showing it is idempotent:

$$\|\_ \lhd \|\_ \lhd U\|\| \quad \subseteq \quad \|\_ \lhd U\|\,,$$

for which we need to prove a lemma stating:

$$\|a \lhd U\| \times \left(\prod_{(u\ :\ |P|)} a' \in U \to \|a' \lhd V\| \right) \to \|a \lhd V\|\,,$$

for every formal topology $\mathcal{F}$ with underlying poset P, $a : |P|$, and downwards-closed subsets $U, V : \mathcal{P}(|P|)$.

The covering nucleus — naive attempt

In the branch case of an attempted proof, the inductive hypothesis gives us

$$\prod_{(c \,:\, C(a,b))} \| d(a,b,c) \lhd V \| ,$$

but what we need is:

$$\left\| \prod_{(c \,:\, C(a,b))} d(a,b,c) \lhd V \right\| .$$

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This inference would require (a form of) the axiom of choice.

In fact, the form of choice needed is provably false [8, Lemma 3.8.5].

The covering nucleus — fixed

As we cannot truncate, we *revise* the inductive definition of $\lhd$ to be a higher inductive type.

$$\frac{a \in U}{a \lhd U}\ \text{dir} \qquad \frac{b : B(a) \qquad \prod_{(c\ :\ C(a,b))} d(a,b,c) \lhd U}{a \lhd U}\ \text{branch}$$

$$\frac{p : a \lhd U \qquad q : a \lhd U}{p = q}\ \text{squash}$$

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The mentioned lemma is now provable without choice *and* the type is propositional!

Generating frames from formal topologies

1. Start with formal topology $\mathcal{F}$ with underlying poset P.
2. Take the frame of downwards-closed subsets of P, denoted $P\downarrow$.
3. $\rhd : P\downarrow \to P\downarrow$ is a nucleus.
4. The generated frame is the frame of fixed points of this nucleus (denoted $\mathfrak{fix}(P\downarrow, \rhd)$).

Formal topologies present

Flat monotonic maps

To state the presentation theorem, we will have to talk about meet-preserving monotonic maps.

However, we are working with posets which may or may not have meets.

The solution is to consider those monotonic maps preserving latent meets: these are called flat monotonic maps.

Let $f : P \to F$ be a **monotonic map** from a poset P to the underlying poset of a frame F. We say that it is flat if:

$$\begin{aligned} \top_F &= \bigvee \{ f(a) \mid a : |P| \}, \quad \text{and} \\ \prod_{(a_0\ a_1\ :\ |P|)} f(a_0) \wedge f(a_1) &= \bigvee \{ f(a_2) \mid a_2 \sqsubseteq a_0 \text{ and } a_2 \sqsubseteq a_1 \}. \end{aligned}$$

Representation

Let

- $\mathcal{F}$ be a formal topology,
- R, a frame, and
- $f : |\mathcal{F}| \to |R|$, a function.

We say that f represents $\mathcal{F}$ in R if:

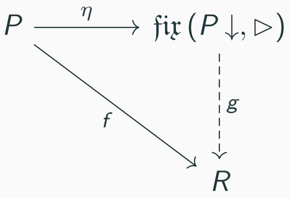
$$\prod_{(a\ :\ A)} \prod_{(b\ :\ B(a))} f(a) \sqsubseteq \bigvee_{c:C(a,b)} f(d(a, b, c)).$$

The main theorem

Theorem. Given

- a formal topology $\mathcal{F}$ with underlying poset P,
- a frame R, and
- a **flat** monotonic map $f : P \to R$;

if f represents $\mathcal{F}$ in R, then there exists a **unique** frame homomorphism g making the following diagram commute:



where $\eta(a) :\equiv \_ \lhd \{a' \mid a' \sqsubseteq a\}$.

Conclusion

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In summary, this thesis development features:

- a reconstruction of the notion of covering within the univalent doctrine as an HIT,
- a *sketch* of the beginnings of an approach for carrying out formal topology in univalent type theory, and
- no postulates, no impredicativity (everything typechecks with `--safe`); no setoids either.

Further work

- Develop more topology using this approach!
- What is the category of formal topologies?
- How can the presentation theorem be stated as an adjunction?

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