Locale Theory in Univalent Foundations

Ayberk Tosun[1]

[1]University of Birmingham, UK

21 March 2023
ASSUME
Nottingham, UK

What is locale theory?

Topology in which the notion of **space** is understood primarily in terms of its

lattice of opens

rather than its

set of points.

What is a locale?

A frame is a lattice

- with finite meets,
- arbitrary joins, and in which
- the meets distribute over the joins.

A locale is a

notion of space characterized by an abstract *frame* of opens.

Lattice theory in UF

The systematic study of lattice theory in UF was started by de Jong and Escardó [JE21; JE23].

They were investigating constructive and predicative UF as a foundational setting for **domain theory**.

Lattice theory in UF

The systematic study of lattice theory in UF was started by de Jong and Escardó [JE21; JE23].

They were investigating constructive and predicative UF as a foundational setting for **domain theory**.

They have implemented several important constructions of domain theory.

- The Scott model of PCF.
- Its soundness and computational adequacy.
- Scott's D_∞ model of the untyped λ-calculus.

Foundational preliminaries – notions of size

Definition ($\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called $\mathcal{V}$**-small** if it has a copy in universe $\mathcal{V}$ i.e. $\sum_{Y:\mathcal{V}} X \simeq Y$.

Foundational preliminaries – notions of size

Definition ($\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **$\mathcal{V}$-small** if it has a copy in universe $\mathcal{V}$ i.e. $\sum_{Y:\mathcal{V}} X \simeq Y$.

Definition (Local $\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **locally $\mathcal{V}$-small** if the identity type $x = y$ is $\mathcal{V}$-small for any two $x, y : X$.

Foundational preliminaries – notions of size

Definition ($\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **$\mathcal{V}$-small** if it has a copy in universe $\mathcal{V}$ i.e. $\sum_{Y:\mathcal{V}} X \simeq Y$.

Definition (Local $\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **locally $\mathcal{V}$-small** if the identity type $x = y$ is $\mathcal{V}$-small for any two $x, y : X$.

Proposition

The following are equivalent.

- For any type $A : \mathcal{U}$, universe $\mathcal{V}$, the type expressing that A is $\mathcal{V}$-small is a proposition.
- The **univalence axiom** holds.

Foundational preliminaries – forms of resizing

Definition (Ω)

We denote by $\Omega_{\mathcal{U}}$ the type of propositions in universe $\mathcal{U}$.

Foundational preliminaries – forms of resizing

Definition (Ω)

We denote by $\Omega_{\mathcal{U}}$ the type of propositions in universe $\mathcal{U}$.

The following was formulated by Voevodsky in [Voe11].

Definition (Propositional resizing)

The **propositional** $(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that every proposition $P : \Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.

Foundational preliminaries – forms of resizing

Definition (Ω)

We denote by $\Omega_{\mathcal{U}}$ the type of propositions in universe $\mathcal{U}$.

The following was formulated by Voevodsky in [Voe11].

Definition (Propositional resizing)

The **propositional** $(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that every proposition $P : \Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.

Definition (Ω-resizing)

The Ω**-**$(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that $\Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.

Foundational preliminaries – forms of resizing

Definition (Ω)

We denote by $\Omega_{\mathcal{U}}$ the type of propositions in universe $\mathcal{U}$.

The following was formulated by Voevodsky in [Voe11].

Definition (Propositional resizing)

The **propositional** $(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that every proposition $P : \Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.

Definition (Ω-resizing)

The Ω**-**$(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that $\Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.

Definition ($\Omega^{\neg\neg}$-resizing)

The $\Omega^{\neg\neg}$**-**$(\mathcal{U}, \mathcal{V})$**-resizing axiom** says that the type $\Omega_{\mathcal{U}}^{\neg\neg}$ is $\mathcal{V}$-small.

Foundational preliminaries – resizing continued

Proposition

Any type $X : \mathcal{U}$ is $(\mathcal{U} \sqcup \mathcal{V})$-small for every universe $\mathcal{V}$.

Foundational preliminaries – resizing continued

Proposition

Any type $X : \mathcal{U}$ is $(\mathcal{U} \sqcup \mathcal{V})$-small for every universe $\mathcal{V}$.

Proposition

Ω-$(\mathcal{U}, \mathcal{V})$-resizing implies propositional $(\mathcal{U}, \mathcal{V})$-resizing.

Foundational preliminaries – resizing continued

Proposition

Any type $X : \mathcal{U}$ is $(\mathcal{U} \sqcup \mathcal{V})$-small for every universe $\mathcal{V}$.

Proposition

Ω-$(\mathcal{U}, \mathcal{V})$-resizing implies propositional $(\mathcal{U}, \mathcal{V})$-resizing.

Proposition

LEM implies Ω-$(\mathcal{U}, \mathcal{V})$-resizing for all universes.

Proof sketch.

- If LEM holds, all propositions are decidable i.e. $\Omega \simeq \mathbf{2}$.
- The type $\mathbf{2}$ always has a copy in $\mathcal{U}_0$.
- Types in $\mathcal{U}_0$ can always be lifted up to any universe.

□

Foundational preliminaries – some basic notions

Definition (Family)

A $\mathcal{U}$**-family on a type** A is a pair (I, α) where $I : \mathcal{U}$ and $\alpha : I \to A$.

We denote the type of $\mathcal{U}$-families on type A by $\mathsf{Fam}_{\mathcal{U}}(A)$ i.e.

$$\mathsf{Fam}_{\mathcal{U}}(A) :\equiv \sum_{I:\mathcal{U}} I \to A.$$

Foundational preliminaries – some basic notions

Definition (Family)

A $\mathcal{U}$**-family on a type** A is a pair (I, α) where $I : \mathcal{U}$ and $\alpha : I \to A$.

We denote the type of $\mathcal{U}$-families on type A by $\mathsf{Fam}_{\mathcal{U}}(A)$ i.e.

$$\mathsf{Fam}_{\mathcal{U}}(A) :\equiv \sum_{I:\mathcal{U}} I \to A.$$

Definition (Directed family)

Let $(x_i)_{i:I}$ be a family on some type A that is equipped with a preorder - $\leq$ -. This family is called **directed** if

1. I is inhabited, and
2. for every $i, j : I$, there exists some $k : I$ such that x_k is an upper bound of $\{x_i, x_j\}$.

Lattices in UF – warm-up

Definition (Join-semilattice)

A $(\mathcal{U}, \mathcal{V})$**-join-semilattice** consists of

- a type $A : \mathcal{U}$,
- a partial order - $\leq$ - : $A \to A \to \Omega_{\mathcal{V}}$,
- a bottom element $\mathbf{0}$: A,
- a binary join operation - $\vee$ - : $A \to A \to A$.

Lattices in UF – warm-up

Definition (Join-semilattice)

A $(\mathcal{U}, \mathcal{V})$**-join-semilattice** consists of

- a type $A : \mathcal{U}$,
- a partial order - $\leq$ - : $A \to A \to \Omega_{\mathcal{V}}$,
- a bottom element $\mathbf{0} : A$,
- a binary join operation - $\vee$ - : $A \to A \to A$.

Proposition

Let A be a $(\mathcal{U}, \mathcal{V})$-join-semilattice. The truth value $x \leq y$ is $\mathcal{W}$-small if and only if the carrier A is a locally $\mathcal{W}$-small type.

Proof sketch.

$(\Rightarrow)$ antisymmetry; $(\Leftarrow)$ $x \leq y \leftrightarrow x \vee y = y$. □

Complete and directed-complete lattices

Definition (Directed-complete poset)

A $\mathcal{W}$**-directed-complete** $(\mathcal{U}, \mathcal{V})$**-poset** consists of

- a type $A : \mathcal{U}$,
- a partial order $- \leq - : A \to A \to \Omega_{\mathcal{V}}$,
- joins of all **directed** $\mathcal{W}$-families.

Complete and directed-complete lattices

Definition (Directed-complete poset)

A $\mathcal{W}$**-directed-complete** $(\mathcal{U},\mathcal{V})$**-poset** consists of

- a type $A : \mathcal{U}$,
- a partial order - $\leq$ - : $A \to A \to \Omega_{\mathcal{V}}$,
- joins of all **directed** $\mathcal{W}$-families.

A dcpo is called **pointed** if it additionally has a least element.

Complete and directed-complete lattices

Definition (Directed-complete poset)

A $\mathcal{W}$**-directed-complete** $(\mathcal{U}, \mathcal{V})$**-poset** consists of

- a type $A : \mathcal{U}$,
- a partial order $\text{-} \leq \text{-} : A \to A \to \Omega_{\mathcal{V}}$,
- joins of all **directed** $\mathcal{W}$-families.

A dcpo is called **pointed** if it additionally has a least element.

Definition (Complete join-lattice)

A $\mathcal{W}$**-complete** $(\mathcal{U}, \mathcal{V})$**-join-lattice** consists of

- a type $A : \mathcal{U}$,
- a partial order $\text{-} \leq \text{-} : A \to A \to \Omega_{\mathcal{V}}$,
- joins of **all** $\mathcal{W}$-families.

Reverse predicative mathematics in UF

Curi [Cur10] previously investigated the limits of predicative mathematics in CZF.

Curi showed:

> *CZF cannot prove that certain classes of nontrivial complete lattices (including join-lattices, dcpos, and frames) are small.*

Reverse predicative mathematics in UF

Curi [Cur10] previously investigated the limits of predicative mathematics in CZF.

Curi showed:

> *CZF cannot prove that certain classes of nontrivial complete lattices (including join-lattices, dcpos, and frames) are small.*

He achieves this by showing that CZF is consistent with an anti-classical principle called GUP.

Reverse predicative mathematics in UF

Curi [Cur10] previously investigated the limits of predicative mathematics in CZF.

Curi showed:

> *CZF cannot prove that certain classes of nontrivial complete lattices (including join-lattices, dcpos, and frames) are small.*

He achieves this by showing that CZF is consistent with an anti-classical principle called GUP.

In contrast to Curi's work, the work of de Jong and Escardó is in the style of **reverse constructive mathematics** [Ish06].

Reverse predicative mathematics in UF

Curi [Cur10] previously investigated the limits of predicative mathematics in CZF.

Curi showed:

> *CZF cannot prove that certain classes of nontrivial complete lattices (including join-lattices, dcpos, and frames) are small.*

He achieves this by showing that CZF is consistent with an anti-classical principle called GUP.

In contrast to Curi's work, the work of de Jong and Escardó is in the style of **reverse constructive mathematics** [Ish06].

They show **directly** that certain results cannot be obtained predicatively, by deriving resizing axioms from them.

Lifting of a proposition

Definition (Lifting)

Let $P : \Omega_{\mathcal{U}}$. The $\mathcal{V}$-lifting of P is defined as

$$\mathcal{L}_{\mathcal{V}}(P) :\equiv \sum_{Q:\Omega_{\mathcal{V}}} Q \to P.$$

Proposition

This is a $\mathcal{V}$-join-lattice when ordered under implication.

A brief summary of de Jong and Escardó's main results

Question

Does $\mathcal{L}_{\mathcal{V}}(P) : \mathcal{V}^{+} \sqcup \mathcal{U}$ have a maximal element?

A brief summary of de Jong and Escardó's main results

Question

Does $\mathcal{L}_{\mathcal{V}}(P) : \mathcal{V}^+ \sqcup \mathcal{U}$ have a maximal element?

Theorem

The following are equivalent.

1. $\mathcal{L}_{\mathcal{V}}(P)$ *has a maximal element.*
2. $\mathcal{L}_{\mathcal{V}}(P)$ *has a greatest monotone inflationary endofunction.*
3. *The identity map* id : $\mathcal{L}_{\mathcal{V}}(P) \to \mathcal{L}_{\mathcal{V}}(P)$ *has a g.f.p.*
4. $\mathcal{L}_{\mathcal{V}}(P)$ *has a small basis.*
5. *Propositional* $(\mathcal{U}, \mathcal{V})$*-resizing holds*

A brief summary of de Jong and Escardó's main results

Question

Does $\mathcal{L}_{\mathcal{V}}(P) : \mathcal{V}^+ \sqcup \mathcal{U}$ have a maximal element?

Theorem

The following are equivalent.

1. $\mathcal{L}_{\mathcal{V}}(P)$ *has a maximal element.*
2. $\mathcal{L}_{\mathcal{V}}(P)$ *has a greatest monotone inflationary endofunction.*
3. *The identity map* id : $\mathcal{L}_{\mathcal{V}}(P) \to \mathcal{L}_{\mathcal{V}}(P)$ *has a g.f.p.*
4. $\mathcal{L}_{\mathcal{V}}(P)$ *has a small basis.*
5. *Propositional* $(\mathcal{U}, \mathcal{V})$*-resizing holds*

Theorem

Existence of a nontrivial complete small poset implies* Ω*-resizing.*

Frames

Definition (Frame)

A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$**-frame** consists of

- a type $A : \mathcal{U}$,
- a partial order - $\leq$ - : $A \to A \to \Omega_{\mathcal{V}}$,
- a top element $\mathbf{1}$: A,
- a binary meet operation - $\wedge$ - : $A \to A \to A$,
- a join operation $\bigvee$ _ : $\mathsf{Fam}_{\mathcal{W}}(A) \to A$;
- satisfying distributivity i.e. $x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$ for every $x : A$ and $\mathcal{W}$-family $(y_i)_{i:I}$ in A.

Some locale theory notation

A **frame homomorphism** is a function preserving finite meets and arbitrary joins.

The category of frames and their homomorphisms is denoted $\mathbf{Frm}$; its opposite is denoted $\mathbf{Loc}$.

- Morphisms of $\mathbf{Loc}$ are called **continuous maps**.

The frame corresponding to a locale X is denoted $\mathcal{O}(X)$.

We work in the spatial direction:

- $X, Y, Z, \ldots$ range over locales;
- $f, g : X \to Y$ range over continuous maps;
- $U, V, W, \ldots : \mathcal{O}(X)$ range over opens; and
- $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ denotes the frame homomorphism corresponding to a continuous map $f : X \to Y$ of locales.

What category of frames do we want?

Question

Do we need all this generality?

No, we don't.

What category of frames do we want?

Question

Do we need all this generality?

No, we don't.

Corollary

If there exists a nontrivial small frame, Ω-resizing holds.

What category of frames do we want?

Question

Do we need all this generality?

No, we don't.

Corollary

If there exists a nontrivial small frame, Ω-resizing holds.

Furthermore, experience shows that most frames that come up in practice are $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-frames ("large and locally small").

What category of frames do we want?

Question

Do we need all this generality?

No, we don't.

Corollary

If there exists a nontrivial small frame, Ω-resizing holds.

Furthermore, experience shows that most frames that come up in practice are $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-frames ("large and locally small").

Accordingly, we restrict attention to large and locally small frames.

Examples of frames

Definition

The **terminal locale** is the locale defined by the frame of opens $\mathcal{O}(\mathbf{1}_{\mathcal{U}}) :\equiv \Omega_{\mathcal{U}}$, ordered under implication. Joins are given by:

$$\left(\bigvee_{i:I} Q_i\right) :\equiv \exists k : I.\, Q_k.$$

Examples of frames

Definition

The **terminal locale** is the locale defined by the frame of opens $\mathcal{O}(\mathbf{1}_{\mathcal{U}}) :\equiv \Omega_{\mathcal{U}}$, ordered under implication. Joins are given by:

$$\left(\bigvee_{i:I} Q_i\right) :\equiv \exists k : I.\, Q_k.$$

Definition

The **discrete locale** over a set X is the type $X \to \Omega_{\mathcal{U}}$ of subsets, ordered under $S \subseteq T :\equiv \forall x : X.\, x \in S \Rightarrow x \in T$, joins given by

$$U \in \left(\bigvee_{i:I} S_i\right) :\equiv \exists i : I.\, U \in S_i.$$

Notion of bases for frames

Definition (Basis)

A family $(B_i)_{i:I}$ of opens forms a **basis** for the frame $\mathcal{O}(X)$ if

for every $U : \mathcal{O}(X)$, there is a **specified**, directed $\mathcal{W}$-family $(i_j)_{j:J}$ on the basis index satisfying $U = \bigvee_{j:J} B_{i_j}$.

Notion of bases for frames

Definition (Basis)

A family $(B_i)_{i:I}$ of opens forms a **basis** for the frame $\mathcal{O}(X)$ if

for every $U : \mathcal{O}(X)$, there is a **specified**, directed $\mathcal{W}$-family $(i_j)_{j:J}$ on the basis index satisfying $U = \bigvee_{j:J} B_{i_j}$.

Definition (Weak basis)

A family $(B_i)_{i:I}$ of opens forms a **weak basis** for $\mathcal{O}(X)$ if

for every open $U : \mathcal{O}(X)$, there is an **unspecified**, directed $\mathcal{W}$-family $(i_j)_{j:J}$ on the basis index satisfying $U = \bigvee_{j:J} B_{i_j}$.

Notion of bases for frames

Definition (Basis)

A family $(B_i)_{i:I}$ of opens forms a **basis** for the frame $\mathcal{O}(X)$ if

for every $U : \mathcal{O}(X)$, there is a **specified**, directed $\mathcal{W}$-family $(i_j)_{j:J}$ on the basis index satisfying $U = \bigvee_{j:J} B_{i_j}$.

Definition (Weak basis)

A family $(B_i)_{i:I}$ of opens forms a **weak basis** for $\mathcal{O}(X)$ if

for every open $U : \mathcal{O}(X)$, there is an **unspecified**, directed $\mathcal{W}$-family $(i_j)_{j:J}$ on the basis index satisfying $U = \bigvee_{j:J} B_{i_j}$.

We use the term basic covering family.

The way-below relation

Definition (The way-below relation)

Open U is said to be **way below** V (denoted $U \ll V$) if

for every directed family $(W_i)_{i:I}$ with $V \leq \bigvee_{i:I} W_i$ there is some $i : I$ with $U \leq W_i$.

Definition (Compact open)

An open $U : \mathcal{O}(X)$ is called **compact** if $U \ll U$

Definition (Compact locale)

A locale X is called **compact** if $\mathbf{1}_X$ is a compact open.

The well-inside relation

Definition (The well-inside relation)

An open U is said to be **well inside** V (denoted $U \lessdot V$) if

there exists some $W : D$ with $U \wedge W = \mathbf{0}_X$ and $V \vee W = \mathbf{1}_X$.

Definition (Clopen)

An open U is called a **clopen** if $U \lessdot U$.

Examples of lattices

In lattice/domain/locale theory, interesting classes of lattices can be defined simply by imposing restrictions on their bases.

Definition (Algebraic dcpo)

A dcpo is called **algebraic** if it has a basis

- that **consists of compact elements**.

Examples of lattices

In lattice/domain/locale theory, interesting classes of lattices can be defined simply by imposing restrictions on their bases.

Definition (Algebraic dcpo)

A dcpo is called **algebraic** if it has a basis

- that **consists of compact elements**.

Definition (Spectral locale)

A locale X is called **spectral** if the frame $\mathcal{O}(X)$ has a small basis

- that **consists of compact elements**, and is
- **closed under finite meets**.

Examples of lattices

In lattice/domain/locale theory, interesting classes of lattices can be defined simply by imposing restrictions on their bases.

Definition (Algebraic dcpo)

A dcpo is called **algebraic** if it has a basis

- that **consists of compact elements**.

Definition (Spectral locale)

A locale X is called **spectral** if the frame $\mathcal{O}(X)$ has a small basis

- that **consists of compact elements**, and is
- **closed under finite meets**.

We refer to such a basis as a spectral basis.

Examples of lattices – continued

Definition (Continuous dcpo)

A dcpo is called **continuous** if it has a basis in which

- the basic covering families **consist of elements way below their joins**.

Examples of lattices – continued

Definition (Continuous dcpo)

A dcpo is called **continuous** if it has a basis in which

- the basic covering families **consist of elements way below their joins**.

Definition (Zero-dimensional locale)

A locale X is called **zero-dimensional** if its frame $\mathcal{O}(X)$ has a basis

- that **consists of clopens**.

Examples of lattices – continued

Definition (Continuous dcpo)

A dcpo is called **continuous** if it has a basis in which

- the basic covering families **consist of elements way below their joins**.

Definition (Zero-dimensional locale)

A locale X is called **zero-dimensional** if its frame $\mathcal{O}(X)$ has a basis

- that **consists of clopens**.

Definition (Regular locale)

A locale X is called **regular** if its frame $\mathcal{O}(X)$ has some basis such that

- the basic covering families **consist of elements well inside their joins**.

Examples of lattices – one final important example

Definition (Stone locale)

A locale X is called **Stone** if its frame $\mathcal{O}(X)$ has a basis

- that consists of opens that are both **compact** and **clopen**,
- is **closed under finite meets**.

Examples of lattices – one final important example

Definition (Stone locale)

A locale X is called **Stone** if its frame $\mathcal{O}(X)$ has a basis

- that consists of opens that are both **compact** and **clopen**,
- is **closed under finite meets**.

Example

The terminal locale $\mathbf{1}_{\mathcal{U}}$ is a Stone locale.
The basis is given by the family $\beta : \mathbf{2} \to \Omega_{\mathcal{U}}$,

$$\beta(0) :\equiv \bot$$
$$\beta(1) :\equiv \top$$

and both of these are clopen and compact.

Spectral locales

Theorem

The following are equivalent for a locale X:

1. *X is spectral i.e. has an unspecified, small spectral basis.*
2. *The type $\mathsf{K}(X)$ of compact opens forms a weak, small spectral basis.*

Spectral locales

Theorem

The following are equivalent for a locale X:

1. *X is spectral i.e. has an unspecified, small spectral basis.*
2. *The type $\mathsf{K}(X)$ of compact opens forms a weak, small spectral basis.*

Notice: the latter is automatically propositional whereas the former has to be truncated.

Spectral locales

Theorem

The following are equivalent for a locale X:

1. *X is spectral i.e. has an unspecified, small spectral basis.*
2. *The type $\mathsf{K}(X)$ of compact opens forms a weak, small spectral basis.*

Notice: the latter is automatically propositional whereas the former has to be truncated.

Corollary

The type of ***specified spectral bases*** *is logically equivalent to its own truncation i.e.* ***has split support*** *[Kra+17].*

Spectral locales

Theorem

The following are equivalent for a locale X:

1. *X is spectral i.e. has an unspecified, small spectral basis.*
2. *The type $\mathsf{K}(X)$ of compact opens forms a weak, small spectral basis.*

Notice: the latter is automatically propositional whereas the former has to be truncated.

Corollary

*The type of **specified spectral bases** is logically equivalent to its own truncation i.e. **has split support** [Kra+17].*

This result seems to use univalence in a crucial way! [WIP]

Stone locales

Definition (Stone locale)

A locale X is called **Stone** if its frame $\mathcal{O}(X)$ has a basis

- that consists of basic opens that are both **compact** and **clopen**, and is
- **closed under binary meets**.

We call this a **Stone basis**.

Stone locales

Definition (Stone locale)

A locale X is called **Stone** if its frame $\mathcal{O}(X)$ has a basis

- that consists of basic opens that are both **compact** and **clopen**, and is
- **closed under binary meets**.

We call this a **Stone basis.**

Corollary

The type of Stone bases has split support.

Stone locales

Definition (Stone locale)

A locale X is called **Stone** if its frame $\mathcal{O}(X)$ has a basis

- that consists of basic opens that are both **compact** and **clopen**, and is
- **closed under binary meets**.

We call this a **Stone basis**.

Corollary

The type of Stone bases has split support.

Proposition

A locale is Stone iff it is compact and zero-dimensional.

Relationship between spectral and Stone locales

Proposition

Every Stone locale is spectral.

Question

What about the other direction?

Relationship between spectral and Stone locales

Proposition

Every Stone locale is spectral.

Question

What about the other direction?

Every spectral locale can be **universally transformed** into a Stone one using the patch topology.

Relationship between spectral and Stone locales

Proposition

Every Stone locale is spectral.

Question

What about the other direction?

Every spectral locale can be **universally transformed** into a Stone one using the patch topology.

$$\mathbf{Stone} \underset{\text{Patch}}{\overset{}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\bot}}}} \mathbf{Spec}$$

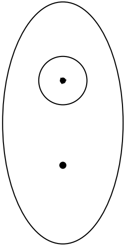
Some examples of patch

Spectral locale in consideration **Its patch**

Some examples of patch

Spectral locale in consideration **Its patch**

Sierpiński space

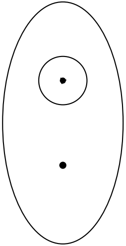
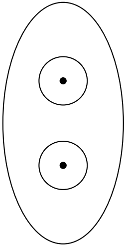


Scott topology of a (Scott) domain

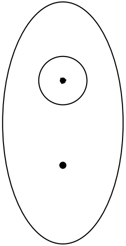
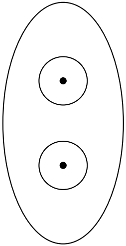
$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$

Scott topology of domain $\mathbb{N}_{\perp}$

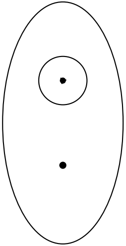
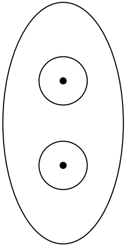
Some examples of patch

Spectral locale in consideration	**Its patch**
Sierpiński space	Booleans ($\mathbf{2}$)
	
Scott topology of a (Scott) domain	
$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$	
Scott topology of domain $\mathbb{N}_{\perp}$	

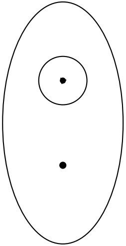
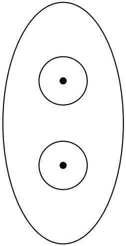
Some examples of patch

Spectral locale in consideration	**Its patch**
Sierpiński space 	Booleans ($\mathbf{2}$) 
Scott topology of a (Scott) domain	Lawson topology
$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$	
Scott topology of domain $\mathbb{N}_\perp$	

Some examples of patch

Spectral locale in consideration	Its patch
Sierpiński space	Booleans ($\mathbf{2}$)
	
Scott topology of a (Scott) domain	Lawson topology
$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$	Cantor space ($\mathbf{2}^{\mathbb{N}}$)
Scott topology of domain $\mathbb{N}_{\perp}$	

Some examples of patch

Spectral locale in consideration	**Its patch**
Sierpiński space	Booleans ($\mathbf{2}$)
	
Scott topology of a (Scott) domain	Lawson topology
$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$	Cantor space ($\mathbf{2}^{\mathbb{N}}$)
Scott topology of domain $\mathbb{N}_\perp$	$\mathbb{N}_\infty$

A description of Patch

A **nucleus** **on locale** X is an endofunction $j : \mathcal{O}(X) \to \mathcal{O}(X)$ that is inflationary, idempotent, and preserves binary meets.

A nucleus is called Scott continuous if it preserves joins of directed families.

A description of Patch

A nucleus on locale X is an endofunction $j : \mathcal{O}(X) \to \mathcal{O}(X)$ that is inflationary, idempotent, and preserves binary meets.

A nucleus is called Scott continuous if it preserves joins of directed families.

Patch is the frame of Scott continuous nuclei.

A description of Patch

A nucleus on locale X is an endofunction $j : \mathcal{O}(X) \to \mathcal{O}(X)$ that is inflationary, idempotent, and preserves binary meets.

A nucleus is called Scott continuous if it preserves joins of directed families.

Patch is the frame of Scott continuous nuclei.

Previous work [Esc99; Esc01] exploited the fact that Patch is a **subframe of the frame of all nuclei**.

Joins in the frame of *all* nuclei (1)

Consider a locale X and let $j, k : \mathcal{O}(X) \to \mathcal{O}(X)$ be two nuclei.

Ordering: $j \preceq k :\equiv \prod_{U:\mathcal{O}(X)} j(U) \leq k(U)$

Top element: $\mathbf{1} :\equiv U \mapsto \mathbf{1}_X$.

Binary meets: $j \curlywedge k :\equiv U \mapsto j(U) \wedge k(U)$.

Joins in the frame of *all* nuclei (1)

Consider a locale X and let $j, k : \mathcal{O}(X) \to \mathcal{O}(X)$ be two nuclei.

Ordering: $j \preceq k :\equiv \prod_{U : \mathcal{O}(X)} j(U) \leq k(U)$

Top element: $\mathbf{1} :\equiv U \mapsto \mathbf{1}_X$.

Binary meets: $j \curlywedge k :\equiv U \mapsto j(U) \wedge k(U)$.

Unfortunately, the **pointwise join** fails to be idempotent in general.

Joins in the frame of *all* nuclei (2)

It is possible to construct the joins in the frame of all nuclei impredicatively i.e. with propositional resizing.

Previous constructions include those by

- Simmons [Sim81],
- Banaschewski [Ban88],
- Johnstone [Joh90],
- Wilson [Wil94], and
- Escardó [Esc03] who uses Pataraia's fixed point theorem.

Joins in the frame of *all* nuclei (2)

It is possible to construct the joins in the frame of all nuclei impredicatively i.e. with propositional resizing.

Previous constructions include those by

- Simmons [Sim81],
- Banaschewski [Ban88],
- Johnstone [Joh90],
- Wilson [Wil94], and
- Escardó [Esc03] who uses Pataraia's fixed point theorem.

Open question

Is the frame of all nuclei **fundamentally impredicative**? If one assumes its existence, *can one derive a form of resizing from it*?

The frame of Scott continuous nuclei

We have constructed [TE23; AET24] the patch locale of a spectral locale and proved the above universal property (completely formalized).

Theorem

*Given any spectral locale X, its **patch locale** $\mathsf{Patch}(X)$, defined by the frame of opens*

$$\mathcal{O}(\mathsf{Patch}(X)) :\equiv \sum_{j:\mathsf{N}(X)} j \textit{ Scott continuous}$$

exhibits $\mathbf{Stone}$ as a coreflective subcategory of $\mathbf{Spec}$.

The construction of the joins goes back to Escardó's previous work [Esc03].

The closed and open nuclei

Theorem

Let X and Y be two locales. If Y has a small, weak basis, then any frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ has right adjoint.

Definition

Let U be an open of a locale X. They Heyting implication $U \Rightarrow (\text{-})$ is defined as the right adjoint of $U \wedge (\text{-})$.

Definition (Open nucleus)

The **open nucleus** on U is defined as $\mathbf{o}(U) :\equiv V \mapsto U \Rightarrow V$.

Definition (Closed nucleus)

The **closed nucleus** on U is defined as $\mathbf{c}(U) :\equiv V \mapsto U \vee V$.

Basis of patch

Theorem

Patch(X) *is large and locally small and has a small basis consisting of clopens.*

Proof sketch

The family

$$\{\mathbf{c}(B_m) \wedge \mathbf{o}(B_n) \mid m, n : I\}$$

forms a basis for Patch(A) and the basic covering family for a given Scott-continuous nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ is given by

$$\{\mathbf{c}(B_m) \wedge \mathbf{o}(B_n) \mid B_m \leq j(B_n), m, n : I\}.$$

References I

[AET24] Igor Arrieta, Martín H. Escardó, and Ayberk Tosun. *The Patch Topology in Univalent Foundations*. 2024. eprint: 2402.03134. URL: https://arxiv.org/abs/2402.03134.

[Ban88] Bernhard Banaschewski. "Another look at the localic Tychonoff theorem". In: *Commentationes Mathematicae Universitatis Carolinae* 29.4 (1988), pp. 647–656.

[Cur10] Giovanni Curi. "On some peculiar aspects of the constructive theory of point-free spaces". In: *Math. Log. Q.* 56.4 (2010), pp. 375–387. DOI: 10.1002/MALQ.200910037. URL: https://doi.org/10.1002/malq.200910037.

[Esc01] Martín H. Escardó. "The regular-locally compact coreflection of a stably locally compact locale". In: *Journal of pure and applied algebra* 157.1 (2001), pp. 41–55. ISSN: 0022-4049.

References II

[Esc03] Martín H. Escardó. "Joins in the frame of nuclei". In: *Applied Categorical Structures* 11.2 (2003), pp. 117–124.

[Esc99] Martín H. Escardó. "On the Compact-regular Coreflection of a Stably Compact Locale". In: vol. 20. 1999, pp. 213–228. DOI: 10.1016/S1571-0661(04)80076-8.

[Ish06] Hajime Ishihara. "Reverse Mathematics in Bishop's Constructive Mathematics". In: *Philosophia Scientiae* CS 6 (2006), pp. 43–59. DOI: 10.4000/philosophiascientiae.406.

[JE21] Tom de Jong and Martín H. Escardó. “Predicative Aspects of Order Theory in Univalent Foundations”. In: *6th International Conference on Formal Structures for Computation and Deduction (FSCD 2021)*. Ed. by Naoki Kobayashi. Vol. 195. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2021, 8:1–8:18. DOI: 10.4230/LIPIcs.FSCD.2021.8.

[JE23] Tom de Jong and Martín Hötzel Escardó. “On Small Types in Univalent Foundations”. In: *Logical Methods in Computer Science* Volume 19, Issue 2 (May 2023). DOI: 10.46298/lmcs-19(2:8)2023. URL: https://lmcs.episciences.org/11270.

[Joh90] Peter T. Johnstone. “Two notes on nuclei”. In: *Order* 7 (1990), pp. 205–210.

References IV

[Kra+17] Nicolai Kraus et al. “Notions of Anonymous Existence in Martin-Löf Type Theory”. In: *Logical Methods in Computer Science* 13.1 (2017). DOI: 10.23638/LMCS-13(1:15)2017.

[Sim81] Harold Simmons. “An algebraic version of Cantor-Bendixson analysis”. In: *Categorical Aspects of Topology and Analysis: Proceedings of an International Conference Held at Carleton University, Ottawa, August 11–15*. Springer. 1981, pp. 310–323.

[TE23] Ayberk Tosun and Martín H. Escardó. “Patch Locale of a Spectral Locale in Univalent Type Theory”. In: *Proceedings of MFPS XXXVIII*. Vol. 1. Electronic Notes in Theoretical Informatics and Computer Science. Feb. 2023. DOI: 10.46298/entics.10808. URL: https://entics.episciences.org/10808.

References V

[Voe11] Vladimir Voevodsky. “Resizing Rules – Their Use and Semantic Justification”. (Bergen, Norway). Sept. 11, 2011. URL: https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/2011_Bergen.pdf. Invited talk at TYPES 2011.

[Wil94] Todd J. Wilson. “The Assembly Tower and Some Categorical and Algebraic Aspects of Frame Theory”. PhD thesis. Pittsburgh, PA: Carnegie Mellon University, 1994.